The identifiability equation follows by a change of variable $t \mapsto 2t/\beta$. □

## 3. Regularization

### 3.1. Preliminaries

**Definition 158** (Operator Fourier transform)**.** *Given a Hamiltonian $H$ and an operator $A$, define the* ***operator Fourier transform (FT)*** *$\hat{A}_H$ by*

$$\hat{A}_H[\omega] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_H(t) e^{-i\omega t} f(t)\, \mathrm{d}t\,,$$

*where $f(t) = e^{-\sigma^2 t^2} \sqrt{\sigma\sqrt{2/\pi}}$ is a Gaussian filter. The "regularizing" role of $f(t)$ will be become clearer in the sequel. Its Fourier transform $\hat{f}[\omega] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, \mathrm{d}t$ satisfies $\hat{f}[\omega] = \frac{1}{\sqrt{\sigma\sqrt{2\pi}}} \exp(-\omega^2/4\sigma^2)$.*

*Note that the operator FT commutes with imaginary time evolution:*

$$e^{\beta H} \hat{A}_H[\omega] e^{-\beta H} = (\widehat{e^{\beta H} A e^{-\beta H}})_H[\omega]$$

Taking the operator FT of both sides of Lemma 150 results in the following useful identity:

$$\hat{A}_H[\omega] = \sum_{\nu} A_\nu \hat{f}[\omega - \nu]\,.$$

In other words, the operator FT gives "soft" access to the components in the Bohr decomposition of $A$. We have a corresponding "soft" Bohr decomposition, by Fourier duality.

**Lemma 159.** *For any operator $A$ and Hermitian $H$,*

$$A = C_\sigma \int_{-\infty}^{\infty} \hat{A}_H[\omega]\, \mathrm{d}\omega$$

*for $C_\sigma := \frac{1}{\sqrt{2\sigma\sqrt{2\pi}}}$.*

Importantly, a straightforward calculation shows that the Gaussian filter ensures the operator FT decays exponentially in the frequency $\omega$:

**Lemma 160.** *For any frequency $\omega$ and operator $A$ satisfying $\|A\|_{\mathsf{op}} \le 1$,*

$$\hat{A}_H[\omega] = e^{-\beta\omega + \sigma^2\beta^2} e^{\beta H} \hat{A}_H[\omega - 2\sigma^2\beta] e^{-\beta H}$$

To see why this is useful, note that because $\|\hat{A}_H[\omega]\|_{\mathsf{op}} \le \hat{f}(0) = O(\sigma^{-1/2})$, this ensures that $\|e^{\beta H} \hat{A}_H[\omega'] e^{-\beta H}\|_{\mathsf{op}} \lesssim e^{\sigma^2\beta^2 + \beta\omega'} \sigma^{-1/2}$. Crucially, the right-hand scales *exponentially in the frequency $\omega'$*, rather than exponentially in the system size! In contrast, norm of the imaginary time-evolved observable $\|e^{\beta H} A e^{-\beta H}\|_{\mathsf{op}}$ can scale exponentially in the system size.

### 3.2. Truncating the identifiability observable

Using Lemma 159, we can decompose $A$ in $\langle O, [A, H - H']\rangle_\rho$ into low-frequency and high-frequency terms under operator FT with respect to $H'$ and apply the identifiability equation in Theorem 153 to obtain

$$\frac{\beta}{2C_\sigma}\langle O,[A,H-H']\rangle_\rho = \int_{|\omega'|\le\Omega'} \mathrm{tr}(\rho\overline{\Delta}[\![H';O,\hat{A}_{H'}[\omega']]\!])\,\mathrm{d}\omega' + \frac{\beta}{2}\int_{|\omega'|\ge\Omega'}\langle O,[\hat{A}_{H'}[\omega'],H-H']\rangle_\rho\,\mathrm{d}\omega'\,.$$

Let us try to write down a slightly more palatable expression for the first integral that doesn't involve the operator FT. Note that

$$\hat{A}_{H'}[\omega']_{H'}(t-i\beta/2) = (\sqrt{\rho'}\hat{A}_{H'}[\omega']\sqrt{\rho'^{-1}})_{H'}(t)\,,$$

and

$$\begin{aligned}\int_{|\omega'|\le\Omega'} &\sqrt{\rho'}\hat{A}_{H'}[\omega']\sqrt{\rho'^{-1}}\,\mathrm{d}\omega' \\ &= \int_{|\omega'|\le\Omega'} \hat{A}_{H'}[\omega'-\sigma^2\beta]e^{-\beta\omega'/2+\sigma^2\beta^2/4}\,\mathrm{d}\omega' \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A_{H'}(t')\underbrace{\int_{|\omega'|\le\Omega'} e^{-i(\omega'-\sigma^2\beta)t'}e^{-\beta\omega'/2+\sigma^2\beta^2/4}\,\mathrm{d}\omega' f(t')}_{h_+(t')}\,\mathrm{d}t'\,,\end{aligned}$$

where in the first step we used Lemma 160, and similarly

$$\begin{aligned}\int_{|\omega'|\le\Omega'} &\sqrt{\rho'^{-1}}\hat{A}_{H'}[\omega']\sqrt{\rho'}\,\mathrm{d}\omega' \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A_{H'}(t')\underbrace{\int_{|\omega'|\le\Omega'} e^{-i(\omega'+\sigma^2\beta)t'}e^{\beta\omega'/2+\sigma^2\beta^2/4}\,D\omega' f(t')}_{h_-(t')}\,\mathrm{d}t'\,.\end{aligned}$$

Observe that

$$|h_+(t)|,|h_-(t)| \le O\Big(\frac{\sqrt{\sigma}}{\beta}e^{-\sigma^2t^2+\beta\Omega'/2+\sigma^2\beta^2/4}\Big)\,,$$

i.e. these functions are rapidly decaying in $t$.

Summarizing, we have the following:

**Lemma 161.** *Let* $\Omega'>0$ *and define the* truncated observable

$$\begin{aligned}&\overline{\Delta}^{\le\Omega'}[\![H';O,A]\!] \\ &:= \frac{1}{\sqrt{2\pi}}\iint_{-\infty}^{\infty}\Big(h_+(t')O_H^\dagger(t)A_{H'}(t'+t) - h_-(t')A_{H'}(t'+t)O_H^\dagger(t)\Big)\,g_\beta(t)\,\mathrm{d}t'\mathrm{d}t\,.\end{aligned} \tag{59}$$

*Then*

$$\frac{\beta}{2C_\sigma}\langle O,[A,H-H']\rangle_\rho = \mathrm{tr}(\rho\overline{\Delta}^{\le\Omega'}[\![H';O,A]\!]) + \frac{\beta}{2}\int_{|\omega'|\ge\Omega'}\langle O,[\hat{A}_{H'}[\omega'],H-H']\rangle_\rho\,\mathrm{d}\omega'\,.$$

Let's take stock of what this buys us. First, because $g_\beta$, $h_+$, and $h_-$ are rapidly decaying, the bulk of the double integral in the truncated observable $\overline{\Delta}^{\le\Omega'}[\![H';O,A]\!]$ is coming from short-time evolutions of $O$ and $A$, which are local by the aforementioned Lieb-Robinson bounds. In short, if we only look at the "low-degree" term in Lemma 161, we now have an observable which is entirely local which captures the discrepancy between $H$ and $H'$.

It still remains to control the truncation error term $\int_{|\omega'|\geq\Omega'}$. For this, we can use Lemma 160 in conjunction with locality of $H - H'$ and $A$ to show that for $\Omega' \geq \Omega(\sigma^2/\mathfrak{d})$, where $\mathfrak{d}$ is the degree of the dual interaction graph, the truncation error is negligible. We defer the details to [**CAN25**, Lemma III.5].

As discussed above, there is still one important missing piece before we can turn the above into a learning algorithm. The issue is that the truncated observable in Eq. (59) ultimately still depends on $H$ through $O^\dagger_H(t)$. We explain the workaround for this next.

## 4. Learning Algorithm

In this section we describe how to exploit the ingredients from the preceding sections, deferring a complete proof of correctness to [**CAN25**].

To sidestep the issue that the truncated observable defined in Eq. (59) depends on $H$, we first define a broader class of observables that contains this observable.

**Definition 162** (General truncated observables)**.** *Fix $\Omega' > 0$. Given operators $K, O, A, G$, with $K$ and $G$ Hermitian, define*

$$\Delta^*[\![G, K; O, A]\!] \coloneqq \frac{1}{\sqrt{2\pi}} \iint_{-\infty}^{\infty} \Big(h_+(t') O^\dagger_G(t) A_K(t'+t) - h_-(t') A_K(t'+t) O^\dagger_G(t)\Big) g_\beta(t)\, \mathrm{d}t' \mathrm{d}t\,. \quad (60)$$

*Note that $\Delta^*[\![H, H'; O, A]\!] = \overline{\Delta}^{\leq\Omega'}[\![H'; O, A]\!]$.*

By design, we have the following:

**Proposition 163.** *When $K = H$, then $\mathrm{tr}(\rho\Delta^*[\![G, K; O, A]\!]) = 0$ for all $O, G, A$.*

Proof. In the proof of Lemma 161, instead of passing to $h_+, h_-$, we can directly express the “low-degree” term $\mathrm{tr}(\rho\overline{\Delta}^{\mathrm{trunc}}(H'; O, A))$ as

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{tr}\Big[\rho \int_{|\omega'|\leq\Omega'} \Big(O^\dagger_H(t) \hat{A}_{H'}[\omega']_{H'}(t+i\beta/2) - \hat{A}_{H'}[\omega']_{H'}(t-i\beta/2) O^\dagger_H(t)\Big)\, \mathrm{d}\omega'\Big]\, \mathrm{d}t\,.$$

In the definition of $\Delta^*[\![G, K; O, A]\!]$, $H'$ and $H$ above are replaced by $K$ and $G$ respectively, yielding

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{tr}\Big[\rho \int_{|\omega'|\leq\Omega'} \Big(O^\dagger_G(t) \hat{A}_K[\omega']_K(t+i\beta/2) - \hat{A}_K[\omega']_K(t-i\beta/2) O^\dagger_H(t)\Big)\, \mathrm{d}\omega'\Big]\, \mathrm{d}t\,.$$

If $K = H$ however, then mirroring the proof of the KMS condition, we have that

$$\mathrm{tr}(\rho O^\dagger_G(t) \hat{A}_H[\omega']_H(t+i\beta/2)) = \mathrm{tr}(\sqrt{\rho} \hat{A}_K[\omega']_H(t) \sqrt{\rho}) = \mathrm{tr}(\rho \hat{A}_H[\omega']_H(t-i\beta/2) O^\dagger_H(t))\,,$$

so $\Delta^*[\![G, H; O, A]\!] = 0$ as claimed. □

This suggests that we can simply brute-force enumerate over a net of different $K$'s, and for each one we check whether $\mathrm{tr}(\rho\Delta^*[\![G, K; O, A]\!]) \approx 0$ for all 1-local Paulis $A$, and $O, G$ in a suitable net. Previously we considered taking $O = [A, H - H']$, but given that this depends on $H$, we can instead use the fact that

$$\|[A, H - H']\|^2_\rho \leq 2\mathfrak{d} \max_a |\langle [A, P_a], [A, H - H'] \rangle_\rho| \quad (61)$$

to restrict to $O = [A, P_a]$ for all 1-local Paulis $A$ and terms $a$ in the support of the Hamiltonian.

The (regularized) identifiability equation in Lemma 161, combined with Lemma 154 and the inequality in Eq. (61), ensures that if $\mathrm{tr}(\rho\Delta^*[\![G, K; O, A]\!]) \approx 0$ for all 1-local $A$, $O = [A, P_a]$, and $G$ in a suitable net, then $K \approx H$.

Only one step remains: how do we enumerate over $G, K$? Naively, if the Hamiltonian has $m$ terms, this would require enumerating over a net over $O(m)$-dimensional parameter space and incurring a runtime scaling exponentially in $O(m)$. Fortunately, there is a workaround that again exploits locality. The intuition is that in the definition of $\Delta^*[\![G, K; O, A]\!]$ in Eq. (60), if $A$ is a 1-local Pauli acting on site $i$, then $A_K(t'+t)$ and $O^\dagger_G(t) = [A, P_a]^\dagger_G(t)$ are roughly supported on a small neighborhood around $i$ (because $t', t$ are not too large because of the exponential damping of $g_\beta, h_+, h_-$). Moreover, Lieb-Robinson bounds ensure that these operators do not change much when $G$ and $K$ are replaced by their truncations to a suitable neighborhood around the $i$-th site. Formally, we have the following estimate:

**Lemma 164** (Lieb-Robinson bound)**.** *If Hamiltonian $H = \sum_a \lambda_a P_a$ with coefficients satisfying $|\lambda_a| \le 1$ has interaction degree $\mathfrak{d}$, then for any operator $A$ acting on subsystem $S \subseteq [n]$ and satisfying $\|A\|_{\mathsf{op}} \le 1$, if $H_\ell$ is given by removing all terms from $H$ at distance at least $\ell$ from $S$, then*

$$\|A_{H_\ell}(t) - A_H(t)\|_{\mathsf{op}} \le O\Big(|S| \cdot \frac{(2\mathfrak{d}|t|)^\ell}{\ell!}\Big)\,.$$

The proof of this will be the subject of one of the homework exercises.
With this in hand, we essentially have a complete, albeit informal, description of the algorithm:

- For each qubit $i \in [n]$:
  (1) Enumerate over a net of local Hamiltonians $K_\ell$ acting on the neighborhood $V(\ell, i)$ of radius $\ell$ around the $i$-th site
  (2) For each such $K_\ell$, use $O(\log n)$ copies of $\rho$ to estimate the observable values $\mathrm{tr}(\rho\Delta^*[\![G_\ell, K_\ell; [A, P_a], A]\!])$ for all local Hamiltonians $G_\ell$ acting on $V(\ell, i)$ and all terms $P_a$ and 1-local Paulis $A$.
  (3) If for any such $K_\ell$ all of these observable values are small, then we will take our estimate of $H$ over the local patch $V(\ell, i)$ to be $K_\ell$.

The quantitative details are somewhat dense and do not provide much additional insight beyond the intuition outlined above, so we defer these to [CAN25].